Basic enumeration problems

Some typical counting problems:

A flag has 3 horizontal stripes. Each stripe is coloured by 9 possible colours. Of course, consecutive stripes cannot have the same colour. How many flags can be made if we respect these rules?

In a class there are 20 pupils. The teacher gives 5 pairwise different books to them for various activities. How many possibilities are there? (Variants: (a) a pupil can only get one book, or (b) can get more.)

In a group of 7 people everyone shakes hands with everyone else (precisely once). What is the total number of handshakes?

What is the number of different hands in the card game bridge?

What is the total number of three-digit numbers in which there are two identical digits?

What is the number five-digit numbers which have a 0 digit? (Variants: they have precisely one (two) 0 digit?; at most one (two)?)

General aim: count **everything**, and count everything **precisely once**.

Basic ideas:

1. Consecutive „independent" decisions
2. We count everything more than once but everything is counted equally often
3. Count the complementary possibilities („Get rid of the bad things")
4. Divide the problem into disjoint subproblems (distinguish cases)

Basic problems:

I. „Ordering" problems (*Permutations*)

1. In how many ways can we arrange n different objects in a list?

Answer: $n \cdot (n-1) \cdot \ldots, 3 \cdot 2 \cdot 1$. This product is denoted as $n!$. We define $0! = 1$.

2. We have objects of r different types. (Objects of the same type are not distinguishable.) There are n_i objects of type i ($i = 1, \ldots r$). In how many ways can we arrange these objects in a list? (In other words, LPV: distributing presents)

Answer: $(n_1 + \ldots n_r)!/(n_1! \ldots n_r!)$

II. „Selection" problems (*Combinations and variations*): we have n different objects and want to choose k of them according to some restrictions:

1. the order of the chosen k object matters and the same object cannot be chosen twice? (in other words: we choose k different objects, one after the other; LPV: number of ordered k-subsets?)

Answer: $n \cdot (n-1) \cdot \ldots (n-k+1) = n!/(n-k)!$

2. the order of the chosen k object does not matter and the same object cannot be chosen twice? (in other words: we choose k different objects at the same time; LPV: number of k-subsets?; lottery tickets)

Answer: $n!/(k!(n-k)!)$. This occurs so often that we denote it by $\binom{n}{k}$. This is called a binomial coefficient.

3. the order of the chosen k object matters and the same object can be chosen twice (repetition is allowed)? (in other words: we choose k not necessarily different objects one after the other; LPV: number of sequences (strings)?; number of subsets)

Answer: $n \cdot n \ldots \cdot n = n^k$.

4. the order of the chosen k object does not matter and the same object can be chosen twice? (in other words: we choose k not necessarily different objects at the same time; icecream balls in a bowl)

Answer: the same as in III.1., that is $\binom{n+k-1}{k}$.

III. A bonus problem

1. Distributing money

a) In how many ways can we distribute k identical (indistinguishable) pieces of coins to n different persons?

Answer: (*Method of stars and bars*

Represent the coins by a star, and use $n-1$ bars separating the coins given to different persons. So: $**||****|***|*|$ means that 10 coins are distributed among 6 people, the first one gets 2, the second 0, the third 4, the fourth 3, the fifth 1, and the sixth person gets 0 coins. There is a bijective map between the distributions and the $*|*$ sequences. In the latter problem we have to order $n-1$ bars and k stars, so this is problem I.2. and the solution $(n+k-1)!/(k!(n-1)!$, which can also be written as $\binom{n+k-1}{k}$.

b) The following variant is even simpler: In how many ways can we distribute k identical (indistinguishable) pieces of coins to n different persons so that everyone gets at least one piece?

In this case we have the same number of stars and bars as before, but in this case we have to put the bars between the coins and two bars cannot be put in the same place. So, we have to choose $n-1$ different places from the $k-1$ places between the stars. This is II.2., so the answer is $\binom{k-1}{n-1}$. If we know the solution of one variant, then we can easily solve the other one.

Be careful, in LPV the letters n, k are just the other way round!!

Tétel. (Omar Khayyam, Binomial Theorem) *For $n \geq 1$ we have*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 \, ldots + \binom{n}{n-1}xy^{n-1}y + \binom{n}{n}y^n,$$

so the coefficient of $x^{n-i}y^i$ in the expansion of $(x+y)^n$ is $\binom{n}{i}$. Using the sum-notation it can be written as

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

How to prove combinatorial identites?

There are two ways, either by algebraic manipulations (e.g. using the definition of binomial coefficients or using the Binomial Theorem) or by inventing a combinatial problem so that it has a simple solution and a complicated one. Since they solve the same problem, they are equal.

A simple example: Prove $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

Algebraic solution:

Put $x = 1 = y$ in the Binomial Theorem.

Combinatorial solution:

An n-element set has 2^n subsets (we have to decise for each elemrnt whether it belongs to the subset or not). On the other hand, there are $\binom{n}{i}$ i-element subsets, so the total number of subsets is equal to theit sum.