Binary GOLAY-codes

$\mathcal{G}_{23}$: $[23,12,7]$ perfect Golay-code

$\mathcal{G}_{24}=\overline{\mathcal{G}_{23}}$ (+pty check bit): $[24,12,8]$

Important properties 1) $\underline{1}\in\mathcal{G}_{23}$ ($\Rightarrow \underline{1}\in\mathcal{G}_{24}$)

2) $\mathcal{G}_{24}^{\perp}=\mathcal{G}_{24}$ (self-dual)

3) $4\,|\,w(\underline{c})$ $\forall\,\underline{c}\in\mathcal{G}_{24}$ (doubly even)

Remark. It is enough to find a basis $\underline{v}_1,\dots \underline{v}_{12}\in\mathcal{G}_{24}$ s.t. $\underline{v}_i\cdot\underline{v}_j=0$ & $4\,|\,w(\underline{v}_i)$

Lemma. $\underline{x},\underline{y}\in\{0,1\}^n$, $4\,|\,w(\underline{x}),w(\underline{y})$. Then: $4\,|\,w(\underline{x}+\underline{y})\Leftrightarrow \underline{x}\cdot\underline{y}=0$.

Pf. Let $X=\mathrm{supp}(\underline{x})$, $Y=\mathrm{supp}(\underline{y})$, $X,Y\subseteq\{1,\dots n\}$. If $|X\cap Y|=c$, then $\underline{x}\cdot\underline{y}=c \pmod 2$, $w(\underline{x}+\underline{y})=|(X\setminus Y)\cup(Y\setminus X)|=|X|+|Y|-2c$.

Icosahedron construction

Consider its edge graph A: adjacency mtx. (12×12 mtx.)

Thm. $G=(I_{12}\,|\,A+J)$ is gen. mtx. of $\mathcal{G}_{24}$ (J: all-1 mtx.)

Pf. Observe that: 1) $\forall$ row of G has wt 8

self dual $\Leftarrow$ 2) rows of G are $\perp$ (2 vertices of icosahedron have 0 or 2 common neighbours $\Rightarrow$ $\forall$ 2 row has 0 or 2 "common "0"-s" $\Rightarrow$ they have 2 or 4 "common "1"-s".

H is also gen. mtx $\Leftarrow$ 3) $H=(A+J\,|\,I_{12})$ is a pty check mtx.

Can $\mathcal{G}_{24}$ have a codeword of $\boxed{\text{wt } 4?}$ $\longleftarrow$ if $\underline{c}\in\mathcal{G}_{24}$ has wt 4, then $\underline{c}$ has either ≤ 2 "1"-s among the first or among the last 12 coordinates. So it is the sum of ≤ 2 rows of G (or H). In both cases the weight is ≥ 8.

Remark. $\exists!$ $(24,2^{12},8)$-code and $\exists!$ $(23,2^{12},7)$-code up to equivalence (= permutation of coordinates).

Ternary GOLAY-codes

$\mathcal{G}_{11}$: $[11, 3^6, 5]_3$ code, over GF(3) (= modulo 3)

$\mathcal{G}_{12} = \overline{\mathcal{G}_{11}}$ (add an extra "bit" s.t. sum of coord.'s becomes "0")

<u>elements</u>: $0, 1, 2$ ($2 = -1$)

We define the gen. mtx.

$$G_{11} = \left(I_6 \;\middle|\; \begin{matrix} 1\;1\;1\;1\;1 \\ S_5 \end{matrix} \right), \text{ where } S_5 = \begin{pmatrix} 0 & + & - & - & + \\ + & 0 & + & - & - \\ - & + & 0 & + & - \\ - & - & + & 0 & + \\ + & - & - & + & 0 \end{pmatrix} \leftarrow 5\times5$$

6x11

$-$ means -1 $(=2)$

$+$ " : 1

quadratic residues (+1), non-residues (−1) mod 5

$$G_{12} = \left(G_{11} \;\middle|\; \begin{matrix} 0 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{matrix} \right)$$

6x12